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Existence Theorems for a Second Order m-Point Boundary Value Problem

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Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and $e \in C[0,1]$. Let $\xi_i \in (0,1)$, $a_i \in \mathbb{R}$, all of the a_i's having the same sign, $i = 1, 2, \ldots, m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ be given. This paper is concerned with the problem of existence of a solution for the m-point boundary value problem

$$x''(t) = f(t, x(t), x'(t)) + e(t), \qquad t \in (0,1) \tag{E}$$

$$x'(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i). \tag{BC$_m$}$$

We discuss existence theorems for the problem (E)–(BC$_m$) under some sign condition of f about the origin. Our analysis is based on a Nonlinear Alternative of Leray-Schauder. © 1997 Academic Press

1. INTRODUCTION

Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous, and $e : [0,1] \to \mathbb{R}$ be a function in $C[0,1]$, $a_i \in \mathbb{R}$, with all of the a_i's having the same sign, $\xi_i \in (0,1)$, $i = 1, 2, \ldots, m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$. The main purpose of this paper is to get results on the solvability of the following boundary value problem (BVP for short)

$$x''(t) = f(t, x(t), x'(t)) + e(t), \qquad t \in (0,1) \tag{E}$$

$$x'(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i). \tag{BC$_m$}$$

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This class of problems falls in the category of what is usually referred to as interior problem or "nonlocal" BVP. See [5, 1] for references along this line.

Very recently, in [3], Gupta considered the problem (E)–(BC_m) under the condition

$$a \overset{\mathrm{df}}{=} \sum_{i=1}^{m-2} a_i \neq 1. \tag{1.1}$$

In this case, the associated linear BVP

$$x'' = 0 \tag{1.2}$$

$$x'(0) = 0, \qquad \sum_{i=1}^{m-2} a_i x(\xi_i) = x(1) \tag{BC_m}$$

has only one solution $x(t) \equiv 0$. In [4], Gupta considered the problem (E)–(BC_m) under the condition

$$a \overset{\mathrm{df}}{=} \sum_{i=1}^{m-2} a_i = 1. \tag{1.3}$$

In this case, the associated BVP (1.2)–(BC_m) has $x(t) = A$, $A \in \mathbb{R}$, as a non-trivial solution. In [3, 4] Gupta established some existence theorems under a growth restriction on f of the form

$$|f(t, x, y)| \leq p(t)|x| + g(t)|y| + r(t), \tag{1.4}$$

where $p, q, r \in L^1[0, 1]$.

The purpose of this paper is to obtain solutions to (E)–(BC_m) under the condition (1.1), or the condition (1.3). We shall replace the growth restriction mentioned above by some sign condition of f about the origin. The sign conditions which are used here were considered in [6] for the classical case of f continuous with two-point boundary conditions and in [9, 10] for the Carathéodory right-hand side f with two-point or functional boundary conditions. Our method is based upon a Nonlinear Alternative of Leray-Schauder [8].

THEOREM (Nonlinear Alternative). *Assume that U is a relatively open subset of a convex set K in a Banach space E. Let $N: \overline{U} \to K$ be a compact map and $p \in U$. Then either*

(i) *N has a fixed point in $\overline{U}$; or*

(ii) *there is a $u \in \partial U$ and $\lambda \in (0, 1)$ such that $u = \lambda N u + (1 - \lambda)p$.*

We shall use the classical spaces $C^k[0,1]$ of all k-times continuously differentiable functions $x:[0,1] \to \mathbb{R}$ with the usual norm

$$\|x\|_k = \max\{\|x\|_0, \|x'\|_0, \ldots, \|x^{(k)}\|_0\}, \tag{1.5}$$

where $\|x\|_0 = \max\{|x(t)| \,|0 \le t \le 1\}$. Let $C^k_{B_0}[0,1]$ be those functions in $C^k[0,1]$ that satisfy the boundary condition (BC_m).

2. THE CASE $a \neq 1$

In this section, we discuss existence theorems under the condition

$$a = \sum_{i=1}^{m-2} a_i \neq 1. \tag{2.1}$$

Define $L_1 : C^2_{B_0}[0,1] \to C[0,1]$ by $Lx = x''$, then from (2.1), we know that L is one to one. Consider the problems

$$x'' = \lambda f(t, x, x') + \lambda e, \qquad 0 < t < 1 \tag{E_λ}$$

$$x'(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i), \tag{BC_m}$$

where $0 < \lambda < 1$. Applying the Nonlinear Alternative we immediately obtain the following

THEOREM 2.1. *Let $a \neq 1$, and let U be an open and bounded neighborhood of $0 \in C^1[0,1]$ such that the problem (E_λ)–(BC_m) has no solutions on the boundary ∂U of U for $0 < \lambda < 1$. Then the problem (E)–(BC_m) has at least one solution in the closure $\overline{U}$ of U.*

Our analysis therefore reduces to constructing an open and bounded set U, such that (E_λ)–(BC_m) has no solutions on ∂U.

It is well known (see [5]) that if a function $x \in C^1[0,1]$ satisfies the boundary condition (BC_m), and $a_i \in \mathbb{R}$, with all of the a_i's having the same sign, then there exists $\eta \in [\xi_1, \xi_{m-2}]$ such that

$$x(1) = ax(\eta), \tag{2.2}$$

where $a = \sum_{i=1}^{m-2} a_i$.

LEMMA 2.2. *Let $\eta \in (0,1)$, $a \in \mathbb{R}$, and $x(t) \in C^1[0,1]$ with $x(1) = ax(\eta)$ be given. Then*

(i) *if $a \le 0$, there exists a $\zeta \in (\eta, 1)$ such that $x(\zeta) = 0$;*

(ii) *if $0 < \alpha \neq 1$, there exists a $\zeta \in (\eta, 1)$ such that $x(\eta) = ((1-\eta)/(\alpha - 1))x'(\zeta)$.*

Proof. See [2, Lemma 22].

THEOREM 2.3. *Let $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ be continuous. Suppose there are constants $L_1, L_2 : L_2 < 0 < L_1$, such that*

$$\text{(i)}\quad f(t,x,L_1)+e(t)\le 0 \qquad \textit{for } (t,x)\in[0,1]\times[-L,L] \tag{2.3}$$

$$\text{(ii)}\quad f(t,x,L_2)+e(t)\ge 0 \qquad \textit{for } (t,x)\in[0,1]\times[-L,L], \tag{2.4}$$

where $L=\max\{L_1,-L_2\}$. Also let $a\le 0$ be given. Then the problem (E)–(BC_m) *has at least one solution.*

Note. The assertion of Theorem 2.3 follows from Corollary 2 in [9].

THEOREM 2.4. *Let $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ be continuous. Suppose there are constants $L_1, L_2 : L_2 < 0 < L_1$ such that*

$$\text{(i)}\quad f(t,x,L_1)+e(t)\le 0 \qquad \textit{for } (t,x)\in[0,1]\times[-\overline{L},\overline{L}] \tag{2.5}$$

$$\text{(ii)}\quad f(t,x,L_2)+e(t)\ge 0 \qquad \textit{for } (t,x)\in[0,1]\times[-\overline{L},\overline{L}], \tag{2.6}$$

where $\overline{L}:\overline{L} > (((1-\xi_1)/(\alpha-1))+1)\max\{-L_2,L_1\}$.
Also let $0<\alpha\neq 1$.
Then the problem (E)–(BC_m) *has at least one solution.*

Proof. By the Tietze-Urysohn Lemma there exists a continuous function $\Delta:\mathbb{R}^2\to[-1,1]$ such that

$$\Delta(x,L_1) = -1, \qquad \text{for } x:|x|\le\overline{L} \tag{2.7}$$

and

$$\Delta(x,L_2) = 1, \qquad \text{for } x:|x|\le\overline{L}. \tag{2.8}$$

For each integer $n\ge 1$ put $f_n(t,x,y)=f(t,x,y)+(1/n)\Delta(x,y)+e(t)$, then

$$f_n(t,x,L_1) < 0 \qquad \text{for } (t,x)\in[0,1]\times\left[-\overline{L},\overline{L}\right] \tag{2.9}$$

$$f_n(t,x,L_2) > 0 \qquad \text{for } (t,x)\in[0,1]\times\left[-\overline{L},\overline{L}\right]. \tag{2.10}$$

Consider the BVP

$$x'' = f_n(t,x,x') \tag{2.11}$$

$$x'(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i). \tag{2.12}$$

The idea is to show that (2.11)–(2.12) has a solution x_n such that

$$-\overline{L}\le x_n\le\overline{L} \qquad \text{and} \qquad L_2\le x_n'\le L_1 \tag{2.13}$$

for all $n \in \mathbb{N}$. Once this is achieved, then by combining (2.13), (2.12), and (2.11) and using the Arzela–Ascoli theorem, we obtain that $\{x_n\}$ has a subsequence which converges in C^2-topology to a solution x_0 for the problem (E)–(BC$_m$).

Define U as the open and bounded neighborhood of $0 \in C^1$ consisting of $v \in C^1$ such that

$$-\overline{L} < v < \overline{L} \qquad \text{and} \qquad L_2 < v' < L_1. \tag{2.14}$$

To prove that (2.11)–(2.12) has a solution x_n such that (2.13) hold, it sufficies to verify, in view of Theorem 2.1, that if $x \in C^1_{B_0}[0,1]$ satisfies

$$-\overline{L} \le x \le \overline{L} \qquad \text{and} \qquad L_2 \le x' \le L_1 \tag{2.15}$$

and

$$x'' = \lambda f_n(t, x, x') \tag{2.16}$$

for some $\lambda \in (0,1)$, then $x \in U$, i.e.,

$$-\overline{L} < x < \overline{L} \qquad \text{and} \qquad L_2 < x' < L_1. \tag{2.17}$$

Now combining the fact that $x \in C^1_{B_0}[0,1]$ and the condition $0 < a \neq 1$ and using Lemma 2.2. we conclude that there is $\zeta \in (\eta, 1)$ such that

$$x(\eta) = \frac{1-\eta}{a-1} x'(\zeta). \tag{2.18}$$

This together with the fact that $L_1 \le x' \le L_2$ and the relation

$$x(t) = x(\eta) + \int_\eta^t x'(s)\, ds \tag{2.19}$$

imply that

$$|x(t)| \le \left|\frac{1-\eta}{a-1}\right| |x'(\xi)| + \left|\int_\eta^t x'(s)\, ds\right| \le \left|\frac{1+\xi_1}{a-1}\right| L + L < \overline{L}. \tag{2.20}$$

Relation (2.20) together with (2.9) and (2.10) implies that

$$f_n(t, x(t), L_1) < 0 \qquad \text{for } t \in [0,1] \tag{2.21}$$

$$f_n(t, x(t), L_2) > 0 \qquad \text{for } t \in [0,1]. \tag{2.22}$$

Suppose that $x'(t_0) = L_1$ for some $t_0 \in [0,1]$. We have $t_0 > 0$ since $x'(0) = 0$. By (2.16) and (2.21), we get that

$$x''(t_0) = \lambda f_n(t_0, x(t_0), x'(t_0)) = \lambda f_n(t_0, x(t_0), L_1) < 0.$$

Therefore, there exists a constant $\delta > 0$, such that

$$x'(t) > L_1 \qquad \text{for } t \in (t_0 - \delta, t_0]. \tag{2.23}$$

On the other hand, (2.15) gives $x'(t) \leq L_1$ for $t \in [0,1]$, a contradiction. Likewise, if there is a constant t_1 such that $x'(t_1) = L_2$, a similar contradiction arises. Thus

$$L_2 < x' < L_1 \qquad \text{for } t \in [0,1]. \tag{2.24}$$

This completes the proof.

3. THE CASE $\alpha = 1$

In this section, we discuss the existence theorem under the condition

$$a = \sum_{i=1}^{m-2} a_i = 1. \tag{3.1}$$

As we have remarked in the Introduction, when $a = 1$, the corresponding linear problem

$$x''(t) = 0, \qquad 0 < t < 1 \tag{1.2}$$

$$x'(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i) \tag{BC$_m$}$$

has $x(t) = A$, $A \in \mathbb{R}$, as a non-trivial solution.

To prove existence under $a = 1$, consider the problems

$$x'' - \varepsilon x' - \varepsilon x = \lambda[f(t,x,x') + e(t) - \varepsilon x' - \varepsilon x] \tag{3.2}$$

$$x'(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i), \tag{3.3}$$

where $\varepsilon > 0$ is a parameter.

Problems (3.2)–(3.3) can be written as

$$Lx = \lambda g(t,x,x') \tag{3.4}$$

$$x \in C^2_{B_0}[0,1], \tag{3.5}$$

where $Lx = x'' - \varepsilon x' - \varepsilon x$, $g(t,x,y) = f(t,x,y) + e(t) - \varepsilon y - \varepsilon x$.

LEMMA 3.1. $L : C^2_{B_0}[0,1] \to C[0,1]$ *is one to one.*

Proof. It is well known (see [5]) that if $x \in C^1[0,1]$ satisfies the boundary condition (BC_m), then there exists $\eta \in [\xi_1, \xi_{m-2}]$ such that

$$x(1) = x(\eta). \tag{3.6}$$

Hence for every solution x of the problem

$$Lx = 0 \tag{3.7}$$

$$x \in C^2_{B_0}[0,1] \tag{3.8}$$

there exists a $\eta \in [\xi_1, \xi_{m-2}]$ such that x is a solution of the three point BVP

$$Lx = 0 \tag{3.9}$$

$$x'(0) = 0, \qquad x(\eta) = x(1). \tag{BC_3}$$

It is easily checked that (3.9)–(BC_3) has no non-trivial solution. Therefore (3.7)–(3.8) has no non-trivial solutions too.

Applying the Nonlinear Alternative, we immediately obtain the following

THEOREM 3.2. *Let* $a = 1$, *and let* U *be an open and bounded neighborhood of* $0 \in C^1[0,1]$ *such that the problem* (3.4)–(3.5) *has no solutions on the boundary* ∂U *of* U *for* $0 < \lambda < 1$. *Then the problem* (E)–(BC_m) *has at least one solution in the closure* $\overline{U}$ *of* U.

THEOREM 3.3. *Let* $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *be continuous and there be* $M > 0$ *and* $\delta > 0$ *such that*

(i) $x[f(t,x,0) + e(t)] \geq \delta > 0$ *for* $|x| > M$

(ii) *there are two constants* $L_1, L_2 : L_1 > M,\ L_2 < -M$ *such that*

$$f(t,x,L_1) + e(x) \geq 0 \qquad \text{for } (t,x) \in [0,1] \times [-M,M] \tag{3.10}$$

$$f(t,x,L_2) + e(x) \leq 0 \qquad \text{for } (t,x) \in [0,1] \times [-M,M] \tag{3.11}$$

(iii) *there are*

$$f(t,x,p) + e(x) < \frac{M}{1-\xi_1}$$
$$\text{for } (t,x,p) \in [\xi_1, 1] \times [-M,M] \times [L_2, L_1] \tag{3.12}$$

$$f(t, x, p) + e(x) > -\frac{M}{1-\xi_1}$$
$$\textit{for } (t, x, p) \in [\xi_1, 1] \times [-M, M] \times [L_2, L_1]. \tag{3.13}$$

Then the problem (E)–(BC_m) *has at least one solution.*

Proof. By the Tietze–Urysohn Lemma there exists a continuous function $\Delta: \mathbb{R}^2 \to [-1, 1]$ such that

$$\Delta(x, L_1) = 1 \qquad \text{for } x \in [-M, M] \tag{3.14}$$

and

$$\Delta(x, L_2) = -1 \qquad \text{for } x \in [-M, M]. \tag{3.15}$$

For each integer $n \geq 1$ put

$$h_n(t, x, y) = f(t, x, y) + e(t) + \frac{1}{n}\Delta(x, y) \tag{3.16}$$

then

$$h_n(t, x, L_1) > 0 \qquad \text{for } (t, x) \in [0, 1] \times [-M, M] \tag{3.17}$$
$$h_n(t, x, L_2) < 0 \qquad \text{for } (t, x) \in [0, 1] \times [-M, M] \tag{3.18}$$

and there is $n_1 \in \mathbb{N}$ such that

$$h_n(t, x, y) < \frac{M}{1-\xi_1}$$
$$\text{for } (t, x, y) \in [\xi_1, 1] \times [-M, M] \times [L_2, L_1] \text{ and } n \geq n_1$$
$$h_n(t, x, y) < \frac{-M}{1-\xi_1}$$
$$\text{for } (t, x, y) \in [\xi_1, 1] \times [-M, M] \times [L_2, L_1] \text{ and } n \geq n_1.$$

Consider the BVP

$$x'' = h_n(t, x, x') \tag{3.19}$$

$$x'(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i). \tag{3.20}$$

The idea is to show that (3.19)–(3.20) has a solution x_n such that

$$-M - 1 \leq x_n \leq M + 1, \qquad L_2 \leq x_n \leq L_1 \tag{3.21}$$

for all $n > n_1$. Once this is achieved, then, be combining (3.21) and (3.19) and using the Arzela–Ascoli theorem, we obtain that $\{x_n\}$ has a subsequence which converges in C^2-topology to a solution of the problem (E)–(BC$_m$).

Take ε to satisfy

$$\varepsilon < \min\left\{1, \frac{\min\{L_1 - M, -M - L_2\}}{\max\{-L_2, L_1\} \cdot (1 - \xi_1)}\right\} \tag{3.22}$$

and define U_1 as the open and bounded neighborhood of $0 \in C^1$ consisting of $v \in C^1$ such that

$$-M - 1 < v < M + 1, \qquad L_2 < v' < L_1. \tag{3.23}$$

To prove (3.19)–(3.20) has a solution x_n such that (3.21) holds, it suffices to verify, in view of Theorem 3.2, that if $x \in C^1_{B_0}[0,1]$ satisfies

$$-M - 1 \le x \le M + 1 \qquad \text{and} \qquad L_2 \le x' \le L_1 \tag{3.24}$$

and

$$x'' = \lambda h_n(t, x, x') + (1 - \lambda)\varepsilon(x' + x) \tag{2.35}$$

for some $\lambda \in (0,1)$, then $x \in U$, i.e.,

$$-M - 1 < x < M + 1, \qquad L_2 < x' < L_1. \tag{3.26}$$

Now let $x \in C^1_{B_0}[0,1]$ satisfy (3.24) and (3.25) for some $\lambda \in (0,1)$. From condition (i), it follows that there exists $n_0 \in \mathbb{N}$, such that $n_0 \ge n_1$ and

$$x\left[\lambda h_n(t, x, 0) + (1 - \lambda)\varepsilon x\right] > 0 \qquad \text{for } |x| > M \text{ and } n > n_0.$$

Consequently, $|x(t)| \le M$ for $t \in [0,1]$, see [2, Lemma 1.2]. Therefore

$$-M - 1 < x(t) < M + 1 \qquad \text{for } t \in [0,1]. \tag{3.27}$$

Relation (3.27) together with (3.17) and (3.18) and the inequalities $L_1 > M$, $L_1 < -M$, implies that

$$\lambda h_n(t, x(t), L_1) + (1 - \lambda)\varepsilon[x(t) + L_1] > 0, \qquad t \in [0,1] \tag{3.28}$$

$$\lambda h_n(t, x(t), L_2) + (1 - \lambda)\varepsilon[x(t) + L_2] < 0, \qquad t \in [0,1] \tag{3.29}$$

for each $\lambda \in (0,1)$.

Suppose that $x'(t_2) = L_1$ for some $t_2 \in [0,1]$. We have $t_2 < 1$. In fact, from condition (iii) and (3.6), we know that there is $\xi \in (\eta, 1) \subset [\xi_1, 1]$, such that $x'(\xi) = 0$. So

$$\begin{aligned} x'(1) &= x'(\xi) + \int_{\xi}^{1} x''(s)\, ds \\ &= \int_{\xi}^{1} \{\lambda f(s, x(s), x'(s)) + (1-\lambda)\varepsilon[x'(s) + x(s)]\}\, ds \\ &\le \frac{\lambda M}{1-\xi_1} \cdot (1-\xi) + (1-\lambda)\varepsilon M(1-\xi_1) \\ &\quad + (1-\lambda)\varepsilon \max\{-L_2, L_1\}(1-\xi_i) \\ &\le M + \varepsilon(1-\xi_1)\max\{-L_2, L_1\} < L_1. \end{aligned} \tag{3.30}$$

From (3.25) and (3.28), it follows that

$$x''(t_2) > 0. \tag{3.31}$$

Therefore there exists $\delta > 0$ such that

$$x'(t) > L_1 \qquad \text{for } t \in (t_2, t_2 + \delta). \tag{3.32}$$

On the other hand, (3.24) gives $x'(t) \le L_1$ for $t \in [0,1]$, a contradiction. Likewise, if there is a constant t_3 such that $x'(t_3) = L_2$, a similar contradiction arises. Thus

$$L_2 < x' < L_1 \qquad \text{for } t \in [0,1]. \tag{3.33}$$

This completes the proof.

4. REMARKS

Remark 4.1. In Theorem 2.4, Theorem 2.3, and Theorem 3.3, the conditions imposed on f *are* local with respect to t, x, and p. But in [3, 4], Gupta discussed existence under a growth restriction on f of the form

$$|f(t,x,y)| \le p(t)|x| + g(t)|y| + r(t), \tag{4.1}$$

where $p, q, r \in L^1[0,1]$.

Remark 4.2. As an application of Theorem 3.3, we mention the boundary value problem

$$x'' = \frac{1}{8} + x + x'^8 \sin(\pi x') \tag{4.2}$$

$$x'(0) = 0, \qquad x(1) = a_1 x\left(\frac{7}{8}\right) + \sum_{i=2}^{m-2} a_i x(\xi_i), \tag{4.3}$$

where $7/8 < \xi_2 < \cdots < \xi_{m-2} < 1$, $a_i > 0$, and $\sum_{i=1}^{m-2} a_i = 1$. Clearly, if we take that $L_1 = 1$, $L_2 = -1$, and $M = 1/4$, then $f(t, x, p) = 1/8 + x + p^8 \sin \pi p, e(t) \equiv 0$ satisfy all assumptions of Theorem 3.3. Therefore (4.2)–(4.3) has at least one solution.

Remark 4.3. Consider the BVP

$$x'' = P_m(x'), \qquad t \in (0,1)$$

$$x'(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i),$$

where $a_i \geq 0$, $i = 1, 2, \ldots, m-2$, $0 < \xi_1 < \cdots < \xi_{m-2} < 1$. If $\sum_{i=1}^{m-2} a_i \neq 1$, and the polynomial $P_m(p)$ has a zero bigger and a zero smaller than 0, then the considered problem is solvable in $C^2[0,1]$; see Theorem 2.4 and Theorem 2.3. It is clear that the results of [3, 4] do not apply to this example.

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